The twin paradox in compact spaces

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Twins travelling at constant relative velocity will each see the other's time dilate leading to the apparent paradox that each twin believes the other ages more slowly. In a finite space, the twins can both be on inertial, periodic orbits so that they have the opportunity to compare their ages when their paths cross. As we show, they will agree on their respective ages and avoid the paradox. The resolution relies on the selection of a preferred frame singled out by the topology of the space.

The twin paradox in special relativity has a simple formulation and resolution in infinite flat space. One twin remains on Earth while the other moves at constant velocity in a spaceship to a distant planet, turns around and returns home to Earth. Each twin believes the other's clock runs slower and so the paradox arises that each believes the other should be younger at their reunion. The paradox is resolved since the twin in the spaceship had to slow down, stop at the distant planet, turn around, and accelerate to constant velocity before returning to Earth. Therefore the travelling twin was not always in an inertial frame and special relativity is not contradicted by the realization that the twin who left Earth is younger than her sibling at the time of their reunion.

In a compact space, the paradox is more complicated. If the travelling twin is on a periodic orbit, she can remain in an inertial frame for all time as she travels around the compact space, never stopping or turning. Since both twins are inertial, both should see the other suffer a time dilation. The paradox again arises that both will believe the other to be younger when the twin in the rocket flies by. The twin paradox can be resolved in compact space and we will show that the twin in the rocket is in fact younger than her sibling after a complete transit around the compact space. The resolution hinges on the existence of a preferred frame introduced by the topology, one consequence of which is the inability of the twin in the rocket to synchronize her clocks [1,2]. While other authors have come to similar conclusions [1–3], the present discussion offers a completely general solution and does not rely on any specific topology. We also make use of the modern language of topology which has recently seen application in cosmology [4].

The manifold of special relativity is $R \otimes \mathcal{M}$ where R represents the time direction and $\mathcal{M} = R^3$ is a flat $3D$ infinite space. The flat spacetime metric is the familiar

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

with $g^\mu_\nu = \mathrm{diag}(-1, 1, 1, 1)$ and

$$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad . \tag{1.2}$$

The spacetime is invariant under the action of the Poincaré group, which contains translations, rotations, and the Lorentz transformations representing relative motion at constant velocity. The isometries can be represented as $O(3,1)$ matrices. We consider special relativity in a compact 3-manifold $\mathcal{M}_c = R \otimes \mathcal{M}/\Gamma$. The elements $\phi \in \Gamma$ act discretely, without fixed points, and are a subset of the full isometry group. The group Γ can be thought of as the set of instructions for compactifying the space. All multiconnected, flat topologies can be constructed from either a parallelepiped or a hexagonal prism with opposite sides identified according to the rules given by the elements $\phi \in \Gamma$ [5–8].

It is advantageous to embed the $(3+1)$-dimensional spacetime in a $(4+1)$-dimensional Minkowski spacetime with the fourth spatial coordinate fixed. Specifically, the $(3+1)$-dimensional coordinate (1.2) is replaced with the $(4+1)$-dimensional coordinate

$$x^a = \begin{pmatrix} t \\ x \\ y \\ z \\ q \end{pmatrix} \tag{1.3}$$

where q is fixed at unity as in fig. 1. We will let Greek indices run over $0, 1, 2, 3$ and Latin indices run over $0, 1, 2, 3, 4$.

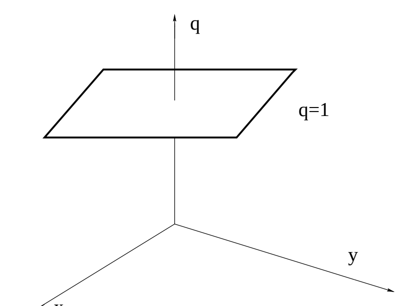

FIG. 1. The embedding of $(3+1)$-Minkowski space into $(4+1)$-Minkowski space. The (t, z) directions are suppressed so that the manifold appears as an infinite sheet fixed at $q = 1$.

In this coordinate system the generators can be represented as 5×5 matrices. For instance, the generator which effects the identification of a point (t, x, y, z, q) with the point $(t, x + L_x, y, z, q)$ can be written as [9]

$$T_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & L_x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.4}$$

so that the boundary condition can be expressed as $x \to T_x x$, which generalizes to

$$x^a \to \phi^a_b x^b \tag{1.5}$$

for each $\phi \in \Gamma$. As an illustration, the hypertorus is constructed by gluing opposite faces of the parallelepiped. The elements of Γ are T_x of eqn (1.4) and

$$T_y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & L_y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, T_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & L_z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Another allowed compact topology is one that first twists the z-faces through π before identification. The elements of Γ for the twisted space are $T_x, T_y, R_z(\pi)T_z$ with $R_z(\theta)$ the rotation matrix:

$$R_z(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.6}$$

All of the multiconnected, flat topologies can be built out of a combination of these translations and rotations.

Periodic orbits are of particular interest since an observer on a periodic orbit can remain inertial. A periodic orbit can be described by the holonomies, $\phi \in \Gamma$, which map the end-point of the orbit to the starting point of the orbit. In other words, a periodic orbit has $x_{\rm end} = \phi x_{\rm start}$ where ϕ can be a composite word $\phi = \prod_i^n \phi_{k_i}$. Each word has a corresponding periodic orbit. For example, consider the periodic orbit of Fig. 2 in the hypertorus. For this orbit we have $x_{\rm end} = T_y T_x^2 x_{\rm start}$.

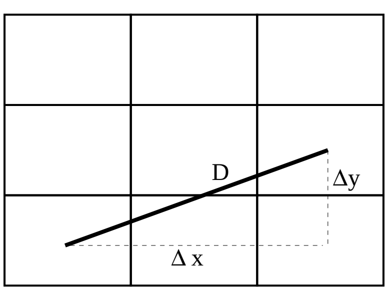

FIG. 2. The compact hypertorus can be represented as an identified parallelepiped. Alternatively, the compact topology can be represented by tiling flat space with identical copies of the fundamental parallelepiped. In the tiling picture above only the (x, y) directions are shown. A particular periodic orbit is drawn which corresponds to $x_{\rm end} = T_y T_x^2 x_{\rm start}$.

Suppose the space is compactified so that with respect to an observer S, only spatial points are identified. In the coordinate system at rest with respect to S, all of the holonomies have $\phi^0_a = 1$. S's twin, H, takes a rocket ride around the compact space, travelling always with constant velocity, never turning, slowing or speeding up (Fig. 3). A coordinate system at rest with respect to H is given by $\bar{x} = \Lambda x$ with Λ the Lorentz transformation. In $(4+1)$ dimensions we can represent the most general Lorentz transformation as

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z & 0 \\ -\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & 0 \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 0 \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the β_i are the velocities of the boosts in the (x, y, z, q) directions, $\beta^2 = \sum_i \beta_i^2$, and $\gamma = 1/\sqrt{1-\beta^2}$. The velocity in the q direction is understood to be zero.

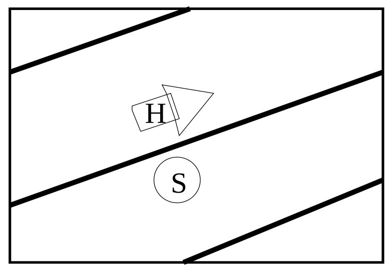

FIG. 3. S stays on Earth while H travels on the periodic orbit of Fig. 2.

S sees H travel a distance D before passing overhead. During one orbital period, S's clock advances a time

$$\Delta t = D/\beta \quad . \tag{1.7}$$

However, S will believe H's clock runs slower by the factor

$$\Delta\bar{t} = \Delta t/\gamma \tag{1.8}$$

and so expects H's clock to advance by

$$\Delta\bar{t} = D/(\gamma\beta) \quad . \tag{1.9}$$

H is therefore younger than S when their paths coincide.

There is no paradox since H will agree that in fact she is younger than her twin. According to H, both space and time points have been identified. As a result it becomes impossible for H to synchronize her clocks [1]. H must be on a periodic orbit to remain inertial. Let H be on a periodic orbit corresponding to the composite word ϕ so the boundary condition (1.5) becomes $\bar{x} = \Lambda x \to \Lambda\phi x$. The lack of synchronicity will be given by the time component of $\Lambda(1-\phi)x$ or explicitly

$$\begin{aligned} \delta\bar{t} &= (\gamma t - \gamma\beta^i x_j) - (\gamma t - \gamma\beta^i \phi_i^j x_j) \\ &= -\gamma\beta^i(x_i - \phi_i^j x_j) \end{aligned} \tag{1.10}$$

with $i = 1, 2, 3, 4$ and the vector $\beta^i = (\beta_x, \beta_y, \beta_z, 0)$, while $x^i = (x, y, z, q)$. The distance travelled as measured by S is $D^2 = \Delta x^a \Delta x_a$ or

$$D = \sqrt{(x_i - \phi_i^j x_j)(x^i - \phi_j^i x^j)} \tag{1.11}$$

and

$$(x_i - \phi_i^j x_j) = D\beta_i/\beta \tag{1.12}$$

so that H's clocks are out of synchronization by a factor

$$\delta\bar{t} = -\gamma\beta D \quad . \tag{1.13}$$

H sees her twin S move away from her in the opposite direction only to return after travelling a distance γD. With the additional time offset of eqn (1.13) due to the compact topology, H's clock must read

$$\Delta\bar{t} = \gamma D/\beta - \gamma\beta D = D/(\gamma\beta) \tag{1.14}$$

in agreement with (1.9). Both twins agree that H is younger than S [10].

Notice that ultimately the age difference between the twins is independent of topology except through the distance D. For the orbit of Fig. 2, for instance, $\phi = T_y T_x^2$ and eqn. (1.11) gives $D = \sqrt{2L_x^2 + L_y^2}$.

The previous example can be recast in a more physical, less abstract discussion. What the above formalism shows is that only one reference frame can be at rest with respect to the compact spatial sections. All other inertial observers in relative motion live in a universe where both space and time points are identified. In the example given around eqns. (1.7)-(1.14), twin S is at rest in a flat torus and H moves inertially along a periodic orbit. Suppose H is initially unaware that spacetime is compact. In order to properly perform any experiments, H has to equip her reference frame with a full system of rulers and clocks. She can set up a system of observers one by one trying to synchronize their clocks by exchanging information with a lightbeam. Somewhere along the way however H will receive her own message telling her to reset her clock by the amount $\gamma\beta D$. She will be out of synch with her own attempts to synchronize. That is, observers at the same spacetime point can have clocks that read different times. H will know that any measurements made in this frame are ambiguous by the time shift.

The twin paradox shows that the compact topology identifies a preferred frame, namely the frame in which the length along a given side is shortest, a point emphasized in Refs. [1] (see also Refs. [2]). To generalize the effect to curved space, Λ can be replaced by an appropriate diffeomorphism and the spacetime topology generalizes to $\mathcal{M}_c = R \otimes \mathcal{M}^U/\Gamma$ where the universal cover, $\mathcal{M}^U$ is a curved, simply connected manifold. Multiconnected cosmologies challenge the Copernican Principle. A compact topology selects a preferred place and a preferred time so that *some* galaxy, if not our own, is at the center of the universe. Some observers are also uniquely able to synchronize their clocks and observe the smallest volume for the universe.

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[9] JL is grateful to Jeff Weeks for pointing out that $3D$ flat space can be embedded in $4D$ to write simple translations as matrices.

[10] The travelling twin must return younger and not older in order to respect the laws of thermodynamics and information flows. Otherwise, the twin could take a black body on a trip around space and return cooler than one on Earth, thereby radiating more quickly than is allowed. (W.H. McCrea, Nature 179, 909 (1957).)
It would be curious if these effects could be exploited to probe extra compact dimensions predicted in some string theories.